AD-A062 410 ROME AIR DEVELOPMENT CENTER GRIFFISS AFB N Y F/G 20/14

NUMERICAL TEST OF KELVIN'S PHASE APPROXIMATION IN CALCULATING T--ETC(U)

APR 78 B PRASAD, E J TICHOVOLSKY

UNCLASSIFIED RADC-TR-78-83 NL

| OF |

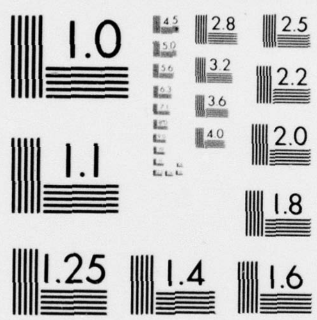
AD A062410

END

DATE FILMED

3-79

DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A062410

DDC FILE COPY

(12) LEVEL II

RADC-TR-78-83
IN-HOUSE REPORT
APRIL 1978

Numerical Test of Kelvin's Phase Approximation in Calculating the Magnetic Field of an Elemental Ring of Meridional Currents on a Sphere

B. PRASAD
E. J. TICHOVOLSKY

ROME AIR DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
GRIFFISS AIR FORCE BASE, NEW YORK 13441

78 12 18 159

This report has been reviewed by the RADC Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be releasable to the general public, including foreign nations.

RADC-TR-78-83 has been reviewed and is approved for publication.

APPROVED: E. A. Lewis

E. A. LEWIS, Chief
Propagation Branch
Electromagnetic Sciences Division

APPROVED: Allan C. Schell

ALLAN C. SCHELL, Acting Chief
Electromagnetic Sciences Division

FOR THE COMMANDER: John P. Huss

JOHN P. HUSS
Acting Chief, Plans Office

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

14

1. REPORT NUMBER: RADC-TR-78-83

2. GOVT ACCESSION NO.:

3. RECIPIENT'S CATALOG NUMBER:

6

4. TITLE (and Subtitle): NUMERICAL TEST OF KELVIN'S PHASE APPROXIMATION IN CALCULATING THE MAGNETIC FIELD OF AN ELEMENTAL RING OF MERIDIONAL CURRENTS ON A SPHERE

5. TYPE OF REPORT & PERIOD COVERED: In-House Report

6. PERFORMING ORG. REPORT NUMBER:

10

7. AUTHOR(s): B. Prasad* E. J. Tichovolsky**

8. CONTRACT OR GRANT NUMBER(s): F30602-75-C-0122

9. PERFORMING ORGANIZATION NAME AND ADDRESS: Deputy for Electronic Technology (RADC/EEP) Hanscom AFB Massachusetts 01731

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 64728F 450A00P1

16 17 00

11. CONTROLLING OFFICE NAME AND ADDRESS: Deputy for Electronic Technology (RADC/EEP) Hanscom AFB Massachusetts 01731

11

12. REPORT DATE: Apri[redacted]978

13. NUMBER OF PAGES: 14

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):

12 14p.

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

*Associated with the RADC Postdoctoral Program

**ARCON Corporation, 260 Bear Hill, Waltham, MA 02154

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

LF ground wave
Ground wave propagation
LORAN
Kelvin's stationary phase principle
Fresnel zone

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An integral expression is derived for the surface magnetic field produced by a ring of meridional currents on a perfectly conducting sphere. This expression is approximated via Kelvin's stationary phase principle, and numerically compared with an evaluation of the integral via Chebyshev's Quadratures. The results have implications for the prediction of LORAN navigation coordinates.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

309 050

Gur

Preface

We would like to thank Dr. E. A. Lewis for suggesting this problem, for providing support, and for valuable discussions. We also would like to thank Mr. J. L. Heckscher for his contributions to these discussions.

This work was funded in part by the Tactical LORAN System Project Office.

Contents

Illustrations

Tables

Numerical Test of Kelvin's Phase Approximation in Calculating the Magnetic Field of an Elemental Ring of Meridional Currents on a Sphere

1. INTRODUCTION

The numerical error introduced by using Kelvin's stationary phase approximation in the calculation of the magnetic field produced by an elemental ring of meridional current moments is examined in this report.[1] Such a calculation arises, for example, in predicting the phase of the LF LORAN signal propagating over a spherical earth.

An exact expression for the magnetic field is derived in the form of a complex integral over the entire ring. This expression then is reduced via Kelvin's approximation to only two contributions: *one from the nearest, the other from the farthest* current moments. The complex integral is numerically integrated via Chebyshev's quadratures and compared with the Kelvin results.

2. THEORETICAL ANALYSIS

Assume a ring of width $\Delta \ell$(m) at the intersection of the sphere $r = a$ with the cone θ = constant and containing a uniform meridional surface current density j_θ (A/m). The directions of the current moments are indicated in Figure 1 by arrows

(Received for publication 7 April 1978)

1. Wintner, A. (1945) Remarks on the method of stationary phases, J. Math. Phys. 24:127-130.

tangent to the sphere along the ring AQ'QBA and pointing toward the pole T. Let the observation point P lie on the sphere at the polar angle θ_o in the meridian plane which passes through the Cartesian x - axis. Then

$$P(x, y, z) = P(a \sin\theta_o, 0, a \cos\theta_o) .$$

A point Q on the moment ring has coordinates

$$Q(x,y,z) = Q(a \sin\theta \cos\phi, a \sin\theta \sin\phi, a \cos\theta) .$$

Consequently, the magnetic field produced at P by the oscillating current moment at Q is given by

$$d\vec{H}(P) = \frac{j_\theta \Delta \ell \, a \sin\theta \, d\phi}{4\pi} \frac{\exp(ikR)}{R} (-ik + \frac{1}{R}) (\hat{\theta} \times \hat{R}) , \tag{1}$$

omitting the spinor exp (-iwt).[2] Here $k = 2\pi/\lambda$ = free space wavenumber;

$$R = a \sqrt{2} (1 - \cos\theta_o \cos\theta - \sin\theta_o \sin\theta \cos\phi)^{1/2}$$

= distance along QP;

$$\hat{R} = \frac{a}{R} [\hat{x}(\sin\theta \cos\phi - \sin\theta_o) + \hat{y} \sin\theta \sin\phi + \hat{z}(\cos\theta - \cos\theta_o)]$$

= unit vector along QP;

$$\hat{\theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z} \sin\theta$$

= unit vector along the current at Q;

and $\hat{x}$, $\hat{y}$, and $\hat{z}$ are Cartesian unit vectors.

The magnetic fields produced by the moments at each pair of conjugate points Q (θ, ϕ) and Q'$(\theta, -\phi)$ cancel in the radial and meridional directions. Thus, the total magnetic field at P is strictly azimuthal and given by

---

2. Stratton, J.A. (1941) Electromagnetic Theory, McGraw-Hill, New York and London, p 435.

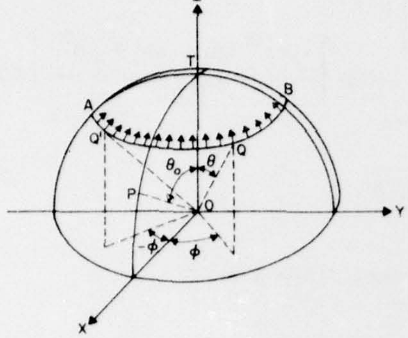


Figure 1. Meridional Current Moments on a Spherical Surface With Cartesian Coordinate Axes at Its Center

$$H_{\phi}(P) = \frac{i k j_{\theta} \, \Delta \ell}{4\pi} \int_0^{2\pi} a \sin\theta \, d\phi \, \frac{\exp(ikR)}{R^2} \left(-ik + \frac{1}{R}\right)$$

$$\cdot \, a(\cos\phi - \sin\theta \, \sin\theta_o - \cos\theta \, \cos\theta_o \cos\phi) \, . \tag{2}$$

Note that the phase factor kR oscillates rapidly when ka is a large number, but that the rest of the integrand is smoothly varying in ϕ. By Kelvin's principle the integral then need be evaluated only in the vicinities of stationary points in the phase factor. Furthermore, the smoothly varying part of the integrand may be treated as if it were constant near any one stationary point and factored out of the integral. The phase factor itself then may be expanded to second order about this particular stationary point and integrated. This procedure must be performed at each stationary point, and then the results are added to obtain the final answer.

The stationary points of kR occur at $\phi = 0$ and π, that is, where

$$\frac{d(kR)}{d\phi} = \frac{ka^2 \sin\theta \, \sin\theta_o \, \sin\phi}{R} = 0 \, .$$

The integral over the expansion of kR about $\phi = 0$ is

$$\int_o^{2\pi} \exp\left[ik R_o + \frac{ik\phi^2 R''_o}{2}\right] d\phi = \exp[ik R_o] \int_o^{2\pi} \exp\left[\frac{ik a^2 \sin\theta \, \sin\theta_o \, \phi^2}{2 R_o}\right] d\phi \, .$$

$$\xrightarrow[a>>1]{} \exp[ik \, R_o + i\pi/4] \sqrt{\frac{\pi \sin\left(\frac{\theta_o - \theta}{2}\right)}{2 \, ak \sin\theta \, \sin\theta_o}} \, , \tag{3}$$

where $R_o = 2a \sin^2 [(\theta_o - \theta)/2]$ and the asymptotic limits of the Fresnel sine and cosine functions were utilized.[3]

3. Gradshteyn, I. S., and Ryzkik, I. M. (1965) Table of Integrals Series and Products, Academic Press, New York and London, pp 930-932.

Similarly for $\phi = \pi$, $R_\pi = 2a \sin^2[(\theta_o + \theta)/2]$ and

$$\int_o^{2\pi} \exp\left[ik R_\pi + \frac{ik\phi^2 R''_\pi}{2}\right] d\phi = \exp[ik R_\pi] \int_o^{2\pi} \exp\left[\frac{-ik a^2 \sin\theta \sin\theta_o \phi^2}{2 R_\pi}\right] d\phi$$

$$\xrightarrow[a>>1]{} \exp[ik R_\pi - i\pi/4] \sqrt{\frac{\sin\left(\frac{\theta_o + \theta}{2}\right)}{2\, ak \sin\theta \sin\theta_o}} . \tag{4}$$

The sum of the contributions from the two stationary points yields

$$H_\phi(P) \approx \frac{i\, j_\theta\, \Delta \ell}{2\lambda} I \tag{5}$$

where I = F - B; here,

$$F = \sqrt{\frac{\lambda}{2a} \frac{\sin\theta}{\sin\theta_o}} \exp\left[i\, 2\, ak \sin\left(\frac{\theta_o - \theta}{2}\right) + \frac{i\pi}{4}\right] \sqrt{\sin\left(\frac{\theta_o - \theta}{2}\right)} \left(1 + \frac{i}{2\, ak \sin\left(\frac{\theta_o - \theta}{2}\right)}\right)$$

arises from the forward moments near $\theta = 0$ while

$$B = \sqrt{\frac{\lambda}{2a} \frac{\sin\theta}{\sin\theta_o}} \exp\left[i\, 2ak \sin\left(\frac{\theta_o + \theta}{2}\right) - \frac{i\pi}{4}\right] \sqrt{\sin\left(\frac{\theta_o + \theta}{2}\right)} \left(1 + \frac{i}{2\, ak \sin\left(\frac{\theta_o + \theta}{2}\right)}\right)$$

arises from the backward moments near $\phi = \pi$. Curves for $|B|$, $|F|$ and $|I|$ vs θ are plotted in Figure 3.

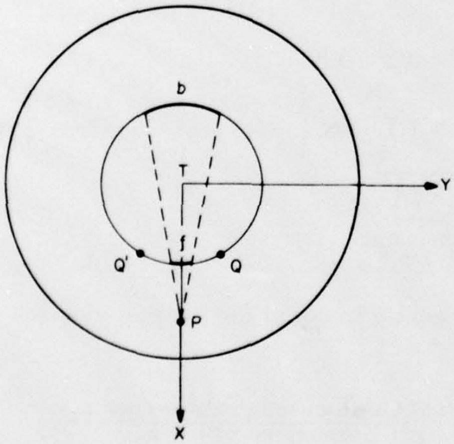


Figure 2. Schematic Representation of the Backward (b) and Forward (f) Fresnel Zones for the Magnetic Field at P Due to Current Dipole Moments Along Ring QQ'Q

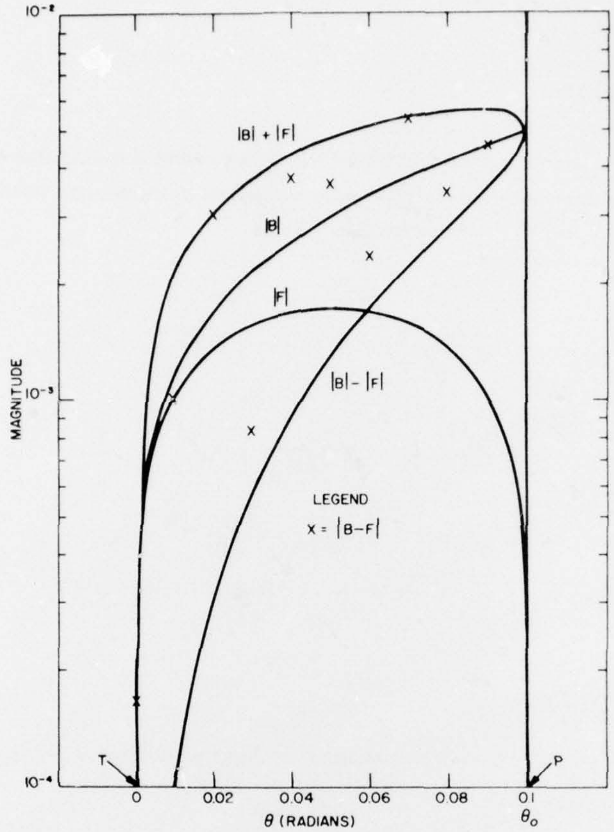


Figure 3. Plot of $|F|$, $|B|$, $|B| + |F|$, $|B| - |F|$, and $|B - F|$ for $0 \leq \theta \leq \theta_o$ Where $\theta_o = 0.1$ radian

Notice that as long as the near field terms are negligible, the contribution from the backward moments is larger than from the forward moments, in the ratio

$$\sqrt{R_\pi / R_o} \doteq \sqrt{\sin\left(\frac{\theta_o + \theta}{2}\right) \bigg/ \sin\left(\frac{\theta_o - \theta}{2}\right)} \; . \qquad (6)$$

This perhaps surprising result arises from the fact that the backward Fresnel zone is larger than the forward zone (as shown in Figure 2) in a way that more than compensates for the additional distance.

The apparent contradiction with Huygen's principle is resolved in a subsequent θ-integration over all such rings for $0 \leq \theta \leq \theta_o$.[4] This integration leads to a second application of Kelvin's principle that eliminates the backward Fresnel zone contribution.

4. Born, M., and Wolf, E. (1970) Principles of Optics, Pergamon Press, Oxford Fourth Edition, pp 370-375.

3. NUMERICAL ANALYSIS

Eq. (2) can be numerically integrated via Chebyshev's quadratures, which is potentially more exact than the Kelvin approximation because the moment ring can be sampled at many points.[5] Rewrite Eq. (2) as

$$H_\phi(P) \approx \frac{i\, j_\theta\, \Delta \ell}{2\lambda}\, I_c\,, \tag{7}$$

where

$$I_c = \frac{\pi}{N} \sum_{n=1}^{N} \frac{\exp(i\sqrt{2}\, a\, k\, \alpha_n)}{\alpha_n^{\,2}} \left(1 + \frac{i}{\sqrt{2}\, a\, k\, \alpha_n}\right) \beta_n\,,$$

$$\alpha_n = (1 - \cos\theta\, \cos\theta_o - \sin\theta\, \sin\theta_o\, \cos\phi_n)^{1/2}\,,$$

$$\beta_n = \sin\theta\, (\cos\phi_n - \sin\theta\, \sin\theta_o - \cos\theta\, \cos\theta_o\, \cos\phi_n)^{1/2}\,,$$

and

$$\phi_n = \pi(2n - 1)/2N\,.$$

I_c was evaluated on a CDC 6600 computer for the following parameters:

$N = 30{,}000$

$a = 6367.951$ km

$\lambda = 3$ km

$\theta_o = 0.1$ rad.

This value of θ_o locates the observation point P at a geodetic distance of 636.8 km from pole T in Figure 1. Table 1 gives the magnitudes of I and I_c, and Table 2 gives the corresponding phases. For $0.01 \le \theta \le 0.09$ there appears a random difference in the magnitudes of I and I_c that is less than 0.6 percent, and a random difference in phase that is less than 0.1 percent (of a maximum possible 2π). This phase difference corresponds to 10 nsec for a 100 kHz LORAN signal.

On the other hand, for $\theta \le 10^{-3}$ rad (that is, for rings less than about 2λ in radius) I and I_c diverge. An approximation of the exact integral, valid for small θ, may be derived from Eq. (2): namely,

$$H_\phi(P) \approx \frac{i\, j_\theta\, \Delta \ell}{2\lambda}\, I'\,, \tag{8}$$

---

5. Abramowitz, M., and Stegun, I.A. (1964) Handbook of Mathematical Functions, National Bureau of Standards Applied Maths. Series, 55, p 889.

where

$$I' = \pi \sin\theta \exp(i\eta)\, J_1(\eta\delta/2)\, [(2/\eta) + i(1 - \delta^2)] ,$$

$$\eta = ak\sqrt{2(1 - \cos\theta \cos\theta_o)} ,$$

$$\delta = \sin\theta \sin\theta_o/(1 - \cos\theta \cos\theta_o) ,$$

and $J_1(x)$ is Bessel's function of first order.

This expression for I' is not derivable from I in Eq. (5). In the second columns of Tables 1 and 2, the values of I' (marked with asterisks) were listed for $\theta \le 10^{-4}$ rad, where there was excellent agreement between I and I'.

Table 1. Comparison of Kelvin's Approximation $|I|$ With Chebyshev's Quadratures $|I_c|$

θ	$\lvert I\rvert$	$\lvert I_c\rvert$	$100(\lvert I\rvert - \lvert I_c\rvert)/\lvert I_c\rvert$
.000001	2.09231×10^{-8}* 1.51515×10^{-5}	2.09231×10^{-8}	0. +70000.
.00001	2.08772×10^{-6}* 4.16843×10^{-5}	2.08772×10^{-6}	0. +2000.
.0001	1.66259×10^{-4}* 1.12897×10^{-4}	1.66136×10^{-4}	+0.07 -32.
.001	2.18594×10^{-5}	4.12690×10^{-6}	+430.
.01	9.87999×10^{-4}	9.93645×10^{-4}	-0.6
.02	3.02082×10^{-3}	3.02006×10^{-3}	+0.03
.03	8.50989×10^{-4}	8.47396×10^{-3}	+0.4
.04	3.73989×10^{-3}	3.74241×10^{-3}	-0.07
.05	3.63608×10^{-3}	3.63226×10^{-3}	+0.1
.06	2.34100×10^{-3}	2.34601×10^{-3}	-0.2
.07	5.31362×10^{-3}	5.31393×10^{-3}	-0.006
.08	3.42070×10^{-3}	3.41286×10^{-3}	+0.2
.09	4.40749×10^{-3}	4.42000×10^{-3}	-0.3
.099	4.98026×10^{-3}	5.01428×10^{-3}	-0.7
.0999	4.71620×10^{-3}	4.82516×10^{-3}	-2.3
.09999	4.93795×10^{-3}	4.82721×10^{-3}	+2.3
.099999	5.25122×10^{-3}	4.59873×10^{-3}	+14.

*$|I'|$ was used for these values.

Table 2. Comparison of Kelvin's Approximation $\angle I$ With Chebyshev's Quadratures $\angle I_c$

θ	$\angle I$	$\angle I_c$	$100(\angle I - \angle I_c)/2\pi$
.000001	5.82201* 2.68117	5.82125	+.01 -50.
.00001	5.82201* 2.68123	5.82125	+.01 -50.
.0001	5.82267* 5.82193	5.82143	+.02 +.008
.001	2.83853	3.66014	-13.
.01	5.70793	5.70768	+.004
.02	5.77161	5.77079	+.01
.03	0.110873	0.113704	-.05
.04	2.29917	2.29834	+.01
.05	2.49314	2.49241	+.01
.06	4.48320	4.48406	-.01
.07	4.99256	4.99130	+.02
.08	5.56764	5.56849	-.01
.09	0.685514	0.685329	+.003
.099	0.600346	0.595509	+.08
.0999	6.20815	6.22394	-.3
.09999	1.16077	1.08156	+1.26
.099999	1.37070	1.18344	+3.00

*$\angle I'$ was used for these values.

4. CONCLUDING REMARKS

From the above it may be concluded that Kelvin's approximation is valid for $2\lambda/a < \theta < \theta_o$ and I' for $\theta < 2\lambda/a$. Near $\theta = \theta_o$ any quantitative comparison is overwhelmed by the divergence caused by the near-field term.